ON THE POWER OF A ONE-SIDED TEST OF FIT AGANIST STOCHASTICALLY COMPARABLE ALTERNATIVES

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I INTRODUCTION

Let F be the continuous cumulative distribution function (c.d.f.) for the random variable (r.v.) X and F_n the empirical c.d.f. determined by the ordered sample $X_1, X_2, \ldots, X_n$ of X. F_n is defined by

$$(1.1) \qquad F_n(x) = \begin{cases} 0 & \text{for } x < X_1 \\ k/n & \text{for } X_k \leq x < X_{k+1};\ k=1,\ldots,n-1 \\ 1 & \text{for } x \geq X_n \end{cases}$$

It is known [1] that the probability $P\{F(x) \leq F_n(x) + \varepsilon, \text{ all } x\}$ is a function independent of F. We will use this function to test the hypothesis $F=H$ against the alternative $F=G$. The power of the test will be studied for alternatives G such that $G(x) \leq H(x)$ for all x and such that

$$\sup_{-\infty < x < -\infty} [H(x)-G(x)] = \delta, \text{ with pre-assigned } \delta > 0.$$

Alternatives of this kind will be called "stochastically comparable, at distance δ from H." For brevity's sake we shall refer to them as alternatives (A).

We assume throughout that $H \in (F)$, $G \in (F)$ where (F) is the set of all continuous strictly increasing c.d.f.'s.

We test $F=H$ against $F=G$ by the following procedure. To have a test of size α for sample size n we will use the value $\varepsilon_{n,\alpha}$ from Table 1 in [2], obtain an ordered sample $X_1, X_2, \ldots, X_n$ of X, determine the empirical c.d.f. of X and reject H if and only if the inequality

$$(1.2) \qquad H(x) < F_n(x) + \varepsilon_{n,\alpha}$$

fails to hold for all real x.

The power of this test is the complementary probability to

(1.3) $$P = P\left\{H(x) < F_n(x) + \varepsilon_{n,\alpha},\ \text{for all } x \,\middle|\, G\right\}.$$

Since (1.2) is satisfied for all x if and only if

(1.4) $$H(X_i) < \frac{i-1}{n} + \varepsilon_{n,\alpha} \text{ for } i=1,\ldots,n$$

we have (writing hereafter ε for $\varepsilon_{n,\alpha}$ and noting that $H(x) \leq 1$ for all x)

$$\begin{aligned} P &= P\left\{H(X_i) < \min\left(\frac{i-1}{n} + \varepsilon, 1\right),\ i=1,\ldots,n \,\middle|\, G\right\} \\ &= P\left\{X_i < \min\left(H^{-1}\left(\frac{i-1}{n} + \varepsilon\right), H^{-1}(1)\right),\ i=1,\ldots,n \,\middle|\, G\right\}. \\ &= P\left\{G(X_i) < \min\left(GH^{-1}\left(\frac{i-1}{n} + \varepsilon\right), GH^{-1}(1)\right),\ i=1,\ldots,n \,\middle|\, G\right\}. \end{aligned}$$

Define the function L by

(1.5) $$L(V) = \begin{cases} \lim\limits_{V \to 0+} GH^{-1}(V) & \text{for } V \leq 0 \\ GH^{-1}(V) & \text{for } 0 < V < 1 \\ \lim\limits_{V \to 1-} GH^{-1}(V) & \text{for } V \geq 1 \end{cases}$$

Recall that $U = G(X)$ has the rectangular distribution R in the unit interval when X has c.d.f. G. We conclude

(1.6) $$P = P\left\{U_i < L\left(\frac{i-1}{n} + \varepsilon\right),\ i=1,\ldots,n \,\middle|\, R\right\}.$$

Clearly $U_1, U_2, \ldots, U_n$ is an ordered sample of the U's. Since the joint probability distribution of $(U_1, U_2, \ldots, U_n)$ is equal to n! for $0 \leq U_1 \leq U_2 \leq \ldots \leq U_n \leq 1$ and zero elsewhere, we have

(1.7) $$P = n! \int_0^{L(\varepsilon)} \int_{U_1}^{L(\frac{1}{n}+\varepsilon)} \cdots \int_{U_{n-1}}^{L(\frac{n-1}{n}+\varepsilon)} dU_n \ldots dU_2\, dU_1.$$

If $H(x) \equiv G(x)$, then $L(V) \equiv V$ for $0 < V < 1$ and (1.7) reduces to formula (3.3) in [2]. If H and G are given, it may be possible to evaluate P, and hence the power $1-P$, from (1.7) by quadrature, or one may compute it by numerical integration. In the general case for alternatives (A) it is possible to derive from (1.7) inequalities for the power, as will be seen.

II ALTERNATIVES (a)

For given hypothesis H, we consider here alternatives G such that $G(x)$ $H(x)$ for all x and such that $\sup_{-\infty<x<\infty} [H(x)-G(x)] = \delta$ for pre-assigned $\delta > 0$. In view of the assumption that $H \in (F)$, $G \in (F)$, the supremum is actually attained, say at X_o. That is,

$$H(X_o)-G(X_o)=\delta.$$

For intuitive reasons one may expect that under these restrictions the power of our test will be close to its minimum when G is close to the function G* defined by

$$(2.1) \qquad G^*(x) = \begin{cases} H(x) & \text{for } x < H^{-1}(U_o) \\ U_o & \text{for } H^{-1}(U_o) \leq x < x_o \\ H(x) & \text{for } x \geq X_o \end{cases}$$

where $U_o = H(X_o) - \delta = G(X_o)$.

To verify this conjecture we consider

$$(2.2) \qquad L^*(V) = G^* H^{-1}(V) = \begin{cases} V & \text{for } 0 < V < U_o \\ U_o & \text{for } U_o \leq V < V_o \\ V & \text{for } V_o \leq V < 1 \\ 1 & \text{for } V \geq 1 \end{cases}$$

where $V_o = H(X_o)$.

Let

$$j = [n(U_o - \varepsilon)]$$
$$(2.3) \qquad k = [n(V_o - \varepsilon)]$$
$$\ell = [n(1 - \varepsilon)]$$

where $[A]$ = greatest integer less than A .

We want to show $L(V) \le L^*(V)$ for all V. We note first that for $0 < V < 1$

$$(2.4) \qquad L(V) \le V.$$

Suppose $X = H^{-1}(V)$. Then $V = H(x)$ and $G(x) = GH^{-1}(V) = L(V)$. By assumption $G(x) \le H(x)$, therefore $L(V) \le V$.

We note next that for $j+1 \le m \le k$ we have $\frac{m}{n} + \varepsilon \le \frac{k}{n} + \varepsilon \le V_o$. Therefore

$$(2.5) \qquad L(\tfrac{m}{n} + \varepsilon) \le L(V_o) = U_o \qquad \text{for } j+1 \le m \le k.$$

Summing up

$$L(V) \le V = L^*(V) \quad \text{for } 0 < V < U_o \text{ and } V_o \le V < 1,$$

which means

$$(2.6) \qquad L(\tfrac{m}{n} + \varepsilon) \le L^*(\tfrac{m}{n} + \varepsilon) \quad \text{for } 0 \le m \le j \text{ and } k+1 \le m \le \ell$$

Further, $L(V) \le 1$ always, so

$$(2.7) \qquad L(\tfrac{m}{n} + \varepsilon) \le 1 = L^*(\tfrac{m}{n} + \varepsilon) \quad \text{for } \ell + 1 \le m.$$

Formulas (2.5), (2.6), (2.7) show us that replacing in (1.7) the function L by the function L^* in the upper limits of integration will

not decrease these limits. Hence

$$(2.8)\quad P \leq n! \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} \int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \int_{U_{\ell+1}}^{1} \cdots \int_{U_{n-1}}^{1}$$

$$dU_n \cdots dU_{\ell+2}\, dU_{\ell+1} \cdots dU_{k+2}\, dU_{k+1} \cdots dU_{j+2}\, dU_{j+1} \cdots dU_2\, dU_1 .$$

Denote the integral on the right by I. We proceed to evaluate I. By (2.1) of 2, we have

$$\int_{U_{\ell+1}}^{1} \cdots \int_{U_{n-1}}^{1} dU_n \cdots dU_{\ell+2} = \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}, \text{ so}$$

$$(2.9)\quad I = \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} \int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}$$

$$dU_{\ell+1} \cdots dU_{k+2}\, dU_{k+1} \cdots dU_{j+2}\, dU_{j+1} \cdots dU_2\, dU_1 .$$

Next we want to evaluate

$$(2.10)\quad \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}\, dU_{\ell+1} \cdots dU_{k+2}$$

Instead we evaluate a slightly more general expression. We shall have occasion to use it later in our discussion as well as now.

Let

$$(2.11)\quad T_j(c,d,\varepsilon) = \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} (c-U_{j+1})^d\, dU_{j+1} \cdots dU_2\, dU_1$$

Then we have

LEMMA 1:

$$(2.12)\quad T_j(c,d,\varepsilon) = \frac{d!}{(d+j+1)!}\left[c^{d+j+1} - \varepsilon\sum_{t=0}^{j}\binom{d+j+1}{t}\left(c-\frac{t}{n}-\varepsilon\right)^{d+j+1-t}\left(\varepsilon+\frac{t}{n}\right)^{t-1}\right]$$

PROOF: By induction on j.

1) For $j = 0$, trivial.

2) Assume the result true for all j, $1 \leq j \leq m$. Then

$$T_{m+1}(c,d,\varepsilon) = \int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}\int_{U_{m+1}}^{\frac{m+1}{n}+\varepsilon}(c-U_{m+2})^{d}\,dU_{m+2}\cdots dU_1$$

$$= \frac{1}{d+1}\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}(c-U_{m+1})^{d+1}\,dU_{m+1}\cdots dU_1$$

$$- \frac{\left(c-\frac{m+1}{n}-\varepsilon\right)^{d+1}}{d+1}\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}dU_{m+1}\cdots dU_1$$

$$= \frac{1}{d+1}T_m(c,d+1,\varepsilon) - \frac{\left(c-\frac{m+1}{n}-\varepsilon\right)^{d+1}}{d+1}\cdot\frac{\varepsilon}{(m+1)!}\left(\varepsilon+\frac{m+1}{n}\right)^{m}$$

(See (2.2) of [2].)

$$T_{m+1}(c,d,\varepsilon) = \frac{1}{d+1}\cdot\frac{(d+1)!}{[(d+1)+m+1]!}\Bigg[c^{(d+1)+m+1} -$$

$$\varepsilon\sum_{t=0}^{m}\binom{(d+1)+m+1}{t}\left(c-\frac{t}{n}-\varepsilon\right)^{(d+1)+m+1-t}\left(\varepsilon+\frac{t}{n}\right)^{t-1}$$

$$-\varepsilon\frac{[(d+1)+m+1]!}{(d+1)!(m+1)!}\left(c-\frac{m+1}{n}-\varepsilon\right)^{d+1}\left(\varepsilon+\frac{m+1}{n}\right)^{m}\Bigg]$$

$$= \frac{d!}{[d+(m+1)+1]!}\Bigg[c^{d+(m+1)+1}-\varepsilon\sum_{t=0}^{m+1}\binom{d+(m+1)+1}{t}\left(c-\frac{t}{n}-\varepsilon\right)^{d+(m+1)+1-t}$$

$$\cdot\left(\varepsilon+\frac{t}{n}\right)^{t-1}\Bigg]$$

This completes the proof of Lemma 1.

Expression (2.10) is evaluated from (2.12) as follows. In (2.10) make the change of variables

$$U_{k+2}-U_{k+1}=v_1$$

(2.13)

$$\vdots$$

$$U_{\ell+1}-U_{k+1}=v_{\ell-k}$$

with Jacobian unity. (2.10) becomes

$$\frac{1}{(n-\ell-1)!}\int_0^{\theta}\int_{v_1}^{\frac{1}{n}+\theta}\cdots\int_{v_{\ell-k-1}}^{\frac{\ell-k-1}{n}+\theta}(1-U_{k+1}-v_{\ell-k})^{n-\ell-1}dv_{\ell-k}\cdots dv_1$$

where $\theta=\frac{k+1}{n}+\varepsilon-U_{k+1}$.

By (2.11) this equals

$$\frac{1}{(n-\ell-1)!}\,T_{\ell-k-1}(1-U_{k+1},n-\ell-1,\varepsilon)=\frac{1}{(n-k-1)!}\Big[(1-U_{k+1})^{n-k-1}-$$

$$\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)\sum_{t=0}^{\ell-k-1}\binom{n-k-1}{t}\left(1-\varepsilon-\frac{k+t+1}{n}\right)^{n-t-k-1}\left(\frac{k+t+1}{n}+\varepsilon-U_{k+1}\right)^{t-1}\Big]$$

$$=\frac{1}{(n-k-1)!}\Big[(1-U_{k+1})^{n-k-1}-\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)\cdot$$

$$\sum_{\nu=k+1}^{\ell}\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu}\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)^{\nu-k-2}\Big].$$

Summarizing our progress to this point, we have

$$I=\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}\frac{(1-U_{k+1})^{n-k-1}}{(n-k-1)!}\,dU_{k+1}\cdots dU_1$$

$$(2.14)\quad -\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}\frac{\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)}{(n-k-1)!}\sum_{\nu=k+1}^{\ell}\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu}.$$

$$\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)^{\nu-k-2}dU_{k+1}\cdots dU_1 .$$

In the second integral in (2.14) write

$\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)$ as $\left(\frac{k+1}{n}-\frac{\nu}{n}\right)+\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)$

and place this factor inside the summation appearing in the integrand.

Then we have

$$I = \frac{1}{(n-k-1)!}\Big\{S(1, n-k-1)$$

$$(2.15) \qquad - \sum_{\nu=k+2}^{\ell} \left(\frac{k+1-\nu}{n}\right)\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu} S\left(\frac{\nu}{n}+\varepsilon, \nu-k-1\right)$$

$$- \sum_{\nu=k+1}^{\ell} \binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu} S\left(\frac{\nu}{n}+\varepsilon, \nu-k-1\right)\Big\}$$

where

$$(2.16) \qquad S(a,b) = \int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}(a-U_{k+1})^b\, dU_{k+1}\cdots dU_1 .$$

We now wish to evaluate $S(a,b)$. To this end let us first examine

$$(2.17) \qquad R_k(a,b) = \int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}(a-U_{k+1})^b\, dU_{k+1}\cdots dU_{j+2} .$$

LEMMA 2:

$$(2.18) \qquad R_k(a,b) = \frac{b!}{(b+k-j)!}\sum_{r=0}^{b}\binom{b+k-j}{r}(a-U_0)^r(U_0-U_{j+1})^{b+k-j-r}$$

PROOF: By induction on k.

1) For $k = j+1$.

$$R_{j+1}(a,b) = \int_{U_{j+1}}^{U_0}(a-U_{j+2})^b\, dU_{j+2}$$

$$= \frac{(a-U_{j+1})^{b+1}}{b+1} - \frac{(a-U_0)^{b+1}}{b+1}$$

The formula yields

$$\frac{b!}{(b+1)!}\sum_{r=0}^{b}\binom{b+1}{r}(a-U_o)^r\,(U_o-U_{j+1})^{b+1-r}$$

$$=\frac{1}{b+1}\left[\sum_{r=0}^{b+1}\binom{b+1}{r}(a-U_o)^r(U_o-U_{j+1})^{b+1-r}-(a-U_o)^{b+1}\right]$$

$$=\frac{1}{b+1}\left[(a-U_{j+1})^{b+1}-(a-U_o)^{b+1}\right].$$

2) Assume the result true for all k, $j+2\leq k\leq m$. Then

$$K_{m+1}(a,b)=\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}\int_{U_{m+1}}^{U_o}(a-U_{m+2})^b\,dU_{m+2}\cdots dU_{j+2}$$

$$=\frac{1}{b+1}\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}(a-U_{m+1})^{b+1}\,dU_{m+1}\cdots dU_{j+2}$$

$$-\frac{(a-U_o)^{b+1}}{b+1}\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}dU_{m+1}\cdots dU_{j+2}$$

$$=\frac{1}{b+1}K_m(a,b+1)-\frac{(a-U_o)^{b+1}}{b+1}\cdot\frac{(U_o-U_{j+1})^{m-j}}{(m-j)!}$$

$$=\frac{1}{b+1}\cdot\frac{(b+1)!}{(b+1+m-j)!}\sum_{r=0}^{b+1}\binom{b+1+m-j}{r}(a-U_o)^r(U_o-U_{j+1})^{b+1+m-j-r}$$

$$-\frac{(a-U_o)^{b+1}}{b+1}\cdot\frac{(U_o-U_{j+1})^{m-j}}{(m-j)!}$$

$$=\frac{b!}{[b+(m+1)-j]!}\sum_{r=0}^{b}\binom{b+(m+1)-j}{r}(a-U_o)^r(U_o-U_{j+1})^{b+(m+1)-j-r}$$

This completes the proof of Lemma 2.

Thus we have now:

$$(2.19) \quad S(a,b) = \frac{b!}{(b+k-j)!} \sum_{r=0}^{b} \binom{b+k-j}{r} (a-U_0)^r \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots$$

$$\cdot \int_{U_j}^{\frac{j}{n}+\varepsilon} (U_0 - U_{j+1})^{b+k-j-r} \, dU_{j+1} \cdots dU_2 \, dU_1 .$$

The integral in the above expression is easily evaluated by Lemma 1. In the notation of the lemma, this integral is equal to

$$T_j(U_0, b+k-j-r) = \frac{(b+k-j-r)!}{(b+k+1-r)!} \Bigg[U_0^{\,b+k+1-r} - \varepsilon \sum_{t=0}^{j} \binom{b+k+1-r}{t} \cdot$$

$$\left(U_0 - \frac{t}{n} - \varepsilon\right)^{b+k+1-r-t} \left(\varepsilon + \frac{t}{n}\right)^{t-1} \Bigg] .$$

Summarizing our results, we obtain

LEMMA 3:

$$(2.20) \quad S(a,b) = \frac{b!}{(b+k+1)!} \sum_{r=0}^{b} \binom{b+k+1}{r} (a-U_0)^r \Bigg[U_0^{\,b+k+1-r} - \varepsilon \sum_{t=0}^{j} \binom{b+k+1-r}{t} \cdot$$

$$\left(U_0 - \frac{t}{n} - \varepsilon\right)^{b+k+1-r-t} \left(\varepsilon + \frac{t}{n}\right)^{t-1} \Bigg] .$$

where $S(a,b)$ is defined by (2.16).

Substituting into (2.15) and after a bit of not difficult algebra, we obtain the

THEOREM: For given hypothesis $H \in (F)$ and for all alternatives $G \in (F)$ such that $G(x) \leq H(x)$ for all x and such that $H(X_0) - G(X_0) = \delta$ for given δ and some X_0, the power of the test described in the introduction is at least as large as $(1 - n!\, I\ )$ where

(2.21)

$$n!\,I = \sum_{r=0}^{n-k-1} \binom{n}{r} (1 - U_o)^r U_o^{\,n-r}$$

$$-\varepsilon \sum_{r=0}^{n-k-1} \sum_{t=0}^{j} \binom{n}{r,t} (1-U_o)^r (U_o - \tfrac{t}{n} - \varepsilon)^{n-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

$$+ \sum_{\nu=k+2}^{\ell} \sum_{r=0}^{\nu-k-2} \binom{n-1}{n-\nu, r} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r U_o^{\nu-1-r}$$

$$-\varepsilon \sum_{\nu=k-2}^{\ell} \sum_{r=0}^{\nu-k-2} \sum_{t=0}^{j} \binom{n-1}{n-\nu, r, t} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r \,\cdot$$

$$(U_o - \tfrac{t}{n} - \varepsilon)^{\nu-1-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

$$- \sum_{\nu=k+1}^{\ell} \sum_{r=0}^{\nu-k-1} \binom{n}{n-\nu, r} (1 - \varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r U_o^{\nu-r}$$

$$+\varepsilon \sum_{\nu=k+1}^{\ell} \sum_{r=0}^{\nu-k-1} \sum_{t=0}^{j} \binom{n}{n-\nu, r, t} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r \,\cdot$$

$$(U_o - \tfrac{t}{n} - \varepsilon)^{\nu-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

and where

$$\binom{n}{a, b, \ldots} \quad \frac{n!}{a!\,b!\ldots(n-a-b-\ldots)!}$$

This lower bound as a function of $H(X_o)$, δ, ε cannot be improved, since for any given $H \in (F)$, X_o, ε, and δ we can construct a $G \in (F)$ arbitrarily close to G^*.

The upper bound for the power is the same as that obtained in [3] although a different alternative is considered there. The upper bound is

$$\sum_{i=0}^{m} \binom{n}{i} \left(1-\varepsilon+\delta-\frac{i}{n}\right)^{n-i}\left(\varepsilon-\delta+\frac{i}{n}\right)^{i-1} \quad \text{for } \varepsilon \geq \delta$$

where $m = [n(1-\varepsilon+\delta)]$ and the upper bound 1 for $\varepsilon < \delta$. This upper bound cannot be improved.

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